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Some Fixed Point Results for Single and Two Maps in 2- Metric Space

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ABSTRACT: In this paper, we establish some fixed point theorems for single and two mappings in 2-metric space which generalize and extend some similar results in the literature.

Keywords: Common fixed points, Metric space, 2-Metric Space, Continuous Function and Cauchy Sequence.

AMS subject classification (2000) – 45H10, 54H25.

I. INTRODUCTION AND PRELIMINARIES

The concept of 2-metric space is a natural generalization of the metric space. Initially, it has been investigated by Gähler [5] and has been developed broadly by Gähler [6, 7] and more. After this number of fixed point theorems have been proved for 2-metric spaces by introducing compatible mappings, which are more general than commuting and weakly commuting mappings. Jungck and Rhoades defined the concepts of d-compatible and weakly compatible mappings as extensions of the concept of compatible mapping for single-valued mappings on metric spaces. Several authors used these concepts to prove some common fixed point theorems. Iseki [10, 11] is well-known in this literature which also include cho et.al., [1,2], Murthy et.al.[15], Naidu and Prasad [16], Pathak et.al. [17]. Vishal Gupta et al [8] also prove some common fixed point theorems for a class of A-contraction on 2- metric space. Various authors [20, 21, 22] used the concepts of weakly commuting mappings, compatible mappings of type (A) and (P) and weakly compatible mappings of type(A) to prove fixed point theorems in 2-metric space. Commutability of two mappings was weakened by Sessa [21] with weakly commuting mappings. Jungck [12] extended the class of non-commuting mappings by compatible mappings.

The purpose of this paper is to establish some fixed point results for single and pair of mappings which generalize and extend some existing well-known results in the literature. Now we start with following definitions, lemmas and theorems.

Definition 1.1: *Let X be a non empty set and d be a real function from $X \times X$ into R^+ such that for all $x, y, z \in X$, we have*

1. $d(x,y) \geq 0$
2. $d(x,y) = 0 \Longrightarrow x = y$
3. $d(x,y) = d(y,x)$
4. $d(x,z) \leq d(x,y) + d(y,z)$

then, d is called a metric or distance function and the pair (X, d) is called a metric space.

Definition 1.2*: A sequence $\{x_n\}$ said to be a Cauchy sequence in 2-metric space X, if for each $a \in X$,*

$$lim_{m,n\to\infty} d(x_n, x, a) = 0$$

Definition 1.3: *A sequence $\{x_n\}$ in 2-metric space X is convergent to an element $x \in X$ if for each $a \in X$,*

$$\lim_{n\to\infty} d(x_n, x, a) = 0$$

Definition 1.4: *A complete 2-metric space is one in which every Cauchy sequence in X converges to an element of X.*

Definition 1.5: *Let A and S be mappings from a metric space (X, d) in to itself, A and S are said to be weakly compatible if they commute at their coincidence point.*

$i.e., Ax = Sx$ for some $x \in X$, then $ASx = SAx$.

Definition 1.6: *Two self maps f and g of a metric space (X, d) are called compatible if*

$$lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X, such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some t in X.

Definition 1.7: *Maps f and g are said to be commuting if $fgx = gfx$ for all $x \in X$*

Definition 1.8: *Let f and g be two self maps on a set X, if $fx = gx$ for some x in X then x is called coincidence point of f and g.*

Throughout this paper X is stand for complete 2-metric space.

Lemma 1.9: *Every subsequence of a convergent sequence to a point x_0 is convergent x_0*

Theorem 1.10 (BANACH'S CONTRACTION MAPPING THEOREM): *Let (X,d) be a complete metric space and $T : X \to X$ be a map such that*

$d(Tx,Ty) \leq \alpha d(x,y)$ *for some* $0 \leq \alpha < 1$ *and all* $x,y \in X$ *then T has a unique fixed point in X. Moreover, for any $x_0 \in X$ the sequence of Picard iterates $\{T^n x_0\}$, $n \geq 0$ converges to the fixed point of T.*

II. MAIN RESULT

Now we prove the following results:

Theorem 2.1:- *Let (X,d) be 2- metric space. Let $T : X \to X$ be continuous mapping satisfying the condition,*

$$\begin{aligned} d(Tx,Ty,a) &\leq \alpha \frac{d(x,Tx,a)d(y,Ty,a) + d(x,Ty,a)d(y,Tx,a)}{d(x,y,a)} \\ &+ \beta \frac{d(x,Ty,a)[d(x,Tx,a)d(y,Ty,a)]}{d(x,y,a) + d(y,Ty,a)d(y,Tx,a)} \\ &+ \gamma \frac{d(x,Tx,a)\, d(y,Tx,a) + d(y,Ty,a)d(x,Ty,a]}{d(x,Tx,a) + d(y,Tx,a) + d(y,Ty,a) + d(x,Ty,a)} \\ &+ \rho \frac{d(x,Tx,a)d(x,Ty,a) + d(y,Ty,a)d(y,Tx,a)}{d(x,Ty,a) + d(y,Tx,a)} \\ &+\delta[d(x,Tx,a) + d(y,Ty,a)] + \eta[d(y,Tx,a) + d(x,Ty,a)] + \mu d(x,y,a) \qquad (1) \end{aligned}$$

for all $x,y \in X, x \neq y$ and for $\alpha,\beta,\gamma,\rho,\delta,\eta,\mu \in [0,1)$ such that $2\alpha + 2\rho + 4\delta + 4\eta + 2\mu < 2$ then T has a unique fixed point in X.

***Proof**. Define $Tx_n = x_{n+1}$ then*

$$d(x_{n+1},x_n,a) = d(Tx_n,Tx_{n-1},a) \leq \alpha \frac{d(x_n,Tx_n,a)d(x_{n-1},Tx_{n-1},a)+d(x_n,Tx_{n-1},a)d(x_{n-1},Tx_n,a)}{d(x_n,x_{n-1},a)}$$

$$+\beta \frac{d(x_n,Tx_{n-1},a)[d(x_n,Tx_n,a)d(x_{n-1},Tx_{n-1},a)]}{d(x_n,x_{n-1},a) + d(x_{n-1},Tx_{n-1},a) + d(x_{n-1},Tx_n,a)}$$

$$+\gamma \frac{d(x_n,Tx_n,a)d(x_{n-1},Tx_n,a) + d(x_{n-1},Tx_{n-1},a)d(x_n,Tx_{n-1},a)}{d(x_n,Tx_n,a) + d(x_{n-1},Tx_n,a) + d(x_{n-1},Tx_{n-1},a) + d(x_n,Tx_{n-1},a)}$$

$$+\rho \frac{d(x_n,Tx_n,a)d(x_n,Tx_{n-1},a) + d(x_{n-1},Tx_{n-1},a)d(x_{n-1},Tx_n,a)}{d(x_n,Tx_{n-1},a) + d(x_{n-1},Tx_n,a)}$$

$$+\delta[d(x_n,Tx_n,a) + d(x_{n-1},Tx_{n-1},a)] + \eta[d(x_{n-1},Tx_n,a) + d(x_n,Tx_{n-1},a) + \mu d(x_n,x_{n-1},a)$$

$$\leq (\alpha + \frac{\gamma}{2} + \delta + \eta)\, d(x_n,x_{n+1},a) + (\rho + \delta + \eta + \mu) d(x_n,x_{n-1},a)$$

$$\therefore d(x_n,x_{n+1},a) \leq \frac{(\rho + \delta + \eta + \mu)}{1 - (\alpha + \frac{\gamma}{2} + \delta + \eta)} d(x_n,x_{n-1},a)$$

Hence, $d(x_{n+1},x_n,a) \leq \lambda d(x_n,x_{n-1},a)$

Where $\lambda = \frac{(\rho+\delta+\eta+\mu)}{1-(\alpha+\frac{\gamma}{2}+\delta+\eta)}$, $\quad 0 \leq \lambda < 1.$

Continuing the same process we get

$$d(x_{n+1},x_n,a) \leq \lambda^n d(x_1,x_0,a)$$

Now for any m, n $(m > n)$ using triangle inequality we have

$$d(x_n, x_m, a) \leq d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+2}, a) + d(x_{n+2}, x_{n+3}, a) + \cdots .. + d(x_{m-1}, x_m, a)$$

$$\leq \lambda^n d(x_1, x_0, a) + \lambda^{n+1} d(x_1, x_0, a) + \lambda^{n+2} d(x_1, x_0, a) + \cdots ... + \lambda^{m-1} d(x_1, x_0, a)$$

$$\leq (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \cdots ... ... + \lambda^{m-1} d(x_1, x_0, a) = \frac{\lambda^n}{1-\lambda} d(x_1, x_0, a)$$

For any $\varepsilon > 0$ *choose a positive number* $N \geq 0$ *such that*

$$\frac{\lambda^n}{1-\lambda} d(x_1, x_0, a) < \varepsilon$$

Then for any $m > n \geq N$,

$$d(x_n, x_m, a) \leq \frac{\lambda^n}{1-\lambda} d(x_1, x_0, a) \leq \frac{\lambda^N}{1-\lambda} d(x_1, x_0, a) < \varepsilon$$

Hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete so there exists a point $u \in X$ such that $x_n \to u$ as $n \to \infty$. Further continuity of T in X implies therefore u is the fixed point of T.

Uniqueness: *If possible, let u and v are two fixed point of T so that by definition we have $Tu = u$ & $Tv = v$. So*

$$d(u, v, a) = d(Tu, Tv, a)$$

$$\leq \alpha \frac{d(u, Tv, a) d(v, Tv, a) + d(u, Tv, a) d(v, Tu, a)}{d(u, v, a)}$$

$$+\beta \frac{d(u,Tv,a)[d(u,Tu,a)+d(v,Tv,a)]}{d(u,v,a)+d(v,Tv,a)+d(v,Tu,a)}$$

$$+\gamma \frac{d(u, Tu, a) d(v, Tu, a) + d(v, Tv, a) d(u, Tv, a)}{d(u, Tu, a) + d(v, Tu, a) + d(v, Tv, a) + d(u, Tv, a)}$$

$$+\rho \frac{d(u,Tu,a)d(u,Tv,a)+d(v,Tv,a)d(v,Tu,a)}{d(u,Tv,a)+d(v,Tu,a)}$$

$$+\delta[d(u, Tu, a) + d(v, Tv, a)] + \eta[d(v, Tu, a) + d(u, Tv, a)]$$

$$+\mu d(u, v, a)$$

which implies

$$d(u, v, a) \leq (\alpha + 2\eta + \mu) d(u, v, a)$$

which is a contradiction,

since $2\alpha + 2\rho + 4\delta + 4\eta + 2\mu < 2$.

Hence $d(u, v, a) = 0 \Rightarrow u = v$.

This completes the proof of the theorem.

Remark: *In theorem (2.1) If*

1. $\alpha = \beta = \gamma = \rho = \delta = \eta = 0$ *then the theorem is reduced to Banach [24]*

2. $\alpha = \beta = \gamma = \rho = \eta = \mu = 0$ *then the theorem is reduced to Kannan [19]*

3. $\alpha = \beta = \gamma = \rho = \eta = 0$ *then the theorem is reduced to Chatterjee [23]*

4. $\alpha = \beta = \gamma = \rho = \delta = 0$ *then the theorem is reduced to Fisher [1]*

5. $\alpha = \beta = \gamma = \rho = 0$ *then the theorem is reduced to Riech [25]*

6. $\alpha = \beta = \gamma = \delta = \eta = \mu = 0$ *then theorem is reduced to M. S. Khan [14]*

7. $\rho = 0$ *then the theorem is reduced to R. Bhardwaj et.al [18]*

Now we establish a result for which T is not necessarily continuous in X but T^r is continuous for some positive integer r then T has a unique fixed point in X.

Theorem 2.2: *Let T be a self mapping defined on 2- metric space (X, d) such that the condition (1) holds. If for some positive integer r, T^r is continuous then T has a unique fixed point in X.*

Proof. *Let us define a sequence* $\{x_n\}$ *as in theorem (2.1), then clearly it converges to some point* u *of* X. *So we can define a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *which also converges to the same point* u *of* X *. Now*

$$T^r u = T^r \left(\lim_{k\to\infty} x_{n_k}\right) = \lim_{k\to\infty}\left(T^r x_{n_k}\right) = \lim_{k\to\infty}\left(x_{n_{k+1}}\right) = u$$

Hence u *is a fixed point of* T^r .
Now we show that $Tu = u$.

Let p *be the smallest positive integer such that* $T^p u = u$ *but*

$T^q \neq u, for\ q = 1,2,3 \ldots\ldots p-1.$

If $p-1 > 0$ *then*

Where $d(Tu,u,a) = d(Tu,T^p u,a) = d(Tu,T(T^{p-1}u),a)$

$$\leq \alpha \frac{d(u,Tu,a)d(T^{p-1}u,T^p u,a) + d(u,T^p u,a)d(T^{p-1}u,T^p u,a)}{d(u,T^{p-1}u,a)}$$

$$+\beta \frac{d(u,T^p u,a)[d(Tu,T^p u,a) + d(T^{p-1}u,T^p u,a)]}{d(u,T^{p-1}u,a)}$$

$$+\gamma \frac{d(u,Tu,a)d(T^{p-1}u,Tu,a) + d(T^{p-1}u,T^p u,a)d(Tu,T^p u,a)}{d(u,Tu,a) + d(T^{p-1}u,Tu,a) + d(T^{p-1}u,T^p u,a) + d(Tu,T^p u,a)}$$

$$+\rho \frac{d(u,Tu,a)d(u,T^p u,a) + d(T^{p-1}u,T^p u,a)d(T^{p-1}u,Tu,a)}{d(u,T^p u,a) + d(T^{p-1}u,Tu,a)}$$

$$+\delta[d(u,Tu,a) + d(T^{p-1}u,T^p u,a)] + \eta[d(T^{p-1}u,Tu,a) + d(u,T^p u,a)] + \mu d(u,T^{p-1}u,a)$$

Such that

$$d(Tu,u,a) \leq \frac{(\delta+\rho+\eta+\mu)}{1-(\alpha+\frac{\gamma}{2}+\delta+\eta)} d(u,T^{p-1}u,a)$$

$$d(u,Tu,a) \leq \lambda d(u,T^{p-1}u,a) \leq \cdots . \leq \lambda^p d(u,Tu,a)$$

where

$$\lambda = \frac{(\delta+\rho+\eta+\mu)}{1-(\alpha+\frac{\gamma}{2}+\delta+\eta)} < 1$$

a contradiction, hence $d(u,Tu,a) = 0 \Rightarrow u = Tu$
The uniqueness can be followed as in theorem (2.1).
This completes the proof of the theorem.

Theorem 2.3: *Let* S *and* T *be mappings of 2- metric space* (X,d) *into itself. Suppose that there exists a non negative real number* α *and* β *such that* $\alpha + 2\beta < 1$ *and*

$$d(Tx,Sy,a) \leq \alpha \frac{d(x,Tx,a)d(x,Sy,a) + d(y,Sy,a)d(y,Tx,a)}{d(x,Sy,a) + d(y,Tx,a)}$$
$$+\beta\, max\{d(x,Tx,a) + d(y,Sy,a), d(y,Sy,a) + d(x,y,a), d(x,Tx,a) + d(x,y,a)\}$$

for all $x,y \in X$ *then* S *and* T *have a unique common fixed point.*

Proof. *Let* $x_0 \in X$ *. Define the sequence* $\{x_n\}$ *by*

$x_{2n+1} = S(x_{2n}), x_{2n+2} = T(x_{2n+1}), n = 0,1,2 \ldots\ldots$ *then we have*

$$d(x, x_2, a) = d(Sx_0, Tx, a) = d(Tx, Sx_0, a)$$

$$\leq \alpha \frac{d(x_1, Tx_1, a)d(x_1, Sx_0, a) + d(x_0, Sx_0, a)d(x_0, Tx_1, a)}{d(x_1, Sx_0, a)d(x_0, Tx_1, a)}$$

$$+\beta max\{d(x_1, Tx_1, a) + d(x_0, Sx_0, a), d(x_0, Sx_0, a)$$
$$d(x_1, x_0, a), d(x_1, Tx_1, a) + d(x_1, x_0, a)\}$$

$$= \alpha \frac{d(x_1, x_2, a)d(x_1, x_1, a) + d(x_0, x_1, a)d(x_0, x_2, a)}{d(x_1, x_1, a) + d(x_0, x_2, a)}$$

$$+\beta max\{d(x_1, x_2, a) + d(x_0, x_1, a), d(x_0, x_1, a) + d(x_1, x_0, a), d(x_1, x_2, a) + d(x_1, x_0, a)\}$$

$$= \alpha d(x_0, x_1, a) + \beta\{d(x_1, x_2, a) + d(x_0, x_1, a)\}$$

$$(1-\beta)d(x_1, x_2, a) \leq (\alpha+\beta)d(x_0, x_1, a)$$

$$d(x_1, x_2, a) \leq \frac{\alpha+\beta}{1-\beta} d(x_0, x_1, a)$$

Put $\lambda = \frac{\alpha+\beta}{1-\beta}$ *where* $0 \leq \lambda < 1$

Then

$$d(x_1, x_2, a) \leq \lambda d(x_0, x_1, a)$$

Similarly we can show,

$$d(x_2, x_3, a) \leq \lambda d(x_1, x_2, a)$$

In general we have

$$d(x_n, x_{n+1}, a) \leq \lambda^n d(x_0, x_1, a)$$

Hence $\{x_n\}$ *ia Cauchy sequence. Since* X *is a 2- metric space, so the sequence* $\{x_n\}$ *converges to some point* x *in* X. *For the point* x,

$$d(x, Tx, a) \leq d(x, x_{n+1}, a) + d(Tx_n, Tx, a)$$

$$= d(x, x_{n+1}, a) + \alpha \frac{d(x_n, Tx_n, a)d(x_n, Tx, a) + d(x, Tx, a)d(x, Tx_n, a)}{d(x_n, Tx, a) + d(x, Tx_n, a)}$$

$$+\beta\, max\{d(x_n, Tx_n, a) + d(x, Tx, a), d(x, Tx) + d(x_n, x), d(x_n, Tx_n, a) + d(x_n, x, a)\}$$

$$= d(x, x_{n+1}, a) + \alpha \frac{d(x_n, x_{n+1}, a)d(x_n, Tx, a) + d(x, Tx, a)d(x, x_{n+1}, a)}{d(x_n, Tx, a) + d(x, x_{n+1}, a)}$$

$$+\beta\, max\{d(x_n, x_{n+1}, a) + d(x, Tx, a), d(x, Tx, a) + d(x_n, x, a), d(x_n, x_{n+1}, a) + d(x_n, x, a)\}$$

Taking limit as $n \to \infty$ *we have,*

$d(x, Tx, a) \leq \beta d(x, Tx, a)$ *a contradiction.*

$\therefore d(x, Tx, a) = 0 \Rightarrow x = Tx.$

Hence x *is the fixed point of* T. *Similarly following the same process we can show that* x *is the fixed point of* S .

Hence x *is the common fixed point of* T *and* S.

Uniqueness: *To show* x *is a unique common fixed point of the mappings* T *and* S *if possible let* y *be a fixed point of* S .

$$d(x, y, a) = d(Tx, Sy, a)$$

$$\leq \alpha \frac{d(x, Tx, a)d(x, Sy, a) + d(y, Sy, a)d(y, Tx, a)}{d(x, Sy, a) + d(y, Tx, a)}$$

$$+\beta\, max\{d(x, Tx, a) + d(y, Sy, a), d(y, Sy, a) + d(x, y, a), d(x, Tx, a) + d(x, y, a)\}$$

$$= \alpha \frac{d(x, x, a)d(x, y, a) + d(y, y, a)d(y, x, a)}{d(x, y, a) + d(y, x, a)}$$

$$+\beta\, max\{d(x, x, a) + d(y, y, a), d(y, y, a) + d(x, y, a), d(x, x, a) + d(x, y, a)\}$$

$$d(x, y, a) \leq \beta d(x, y, a)$$

which is a contradiction, since $\alpha + 2\beta < 1$, *Hence* $d(x, y, a) = 0 \Rightarrow x = y$. *This completes the proof of the theorem.*

Remark: *If* $\beta = 0$ *we get theorem (2.1) of M.S. Khan [14]*

If $\alpha = 0$ *we get theorem 2.2 of R. Shrivastva et. al. [20].*

If $S = T$ *then we get the following*

Corrollary 2.4:

$d(Tx, Ty, a)$

$$+\beta\max\{d(x,Tx,a)+d(y,Ty,a),d(y,Ty,a)+d(x,y,a),d(x,Tx,a)+d(x,y,a)$$

Remark: *If* $\alpha = 0$ *then we get A -Contraction introduced by M. Akram et.al [13].*
If $\beta = 0$ *we get theorem (1.10) of M. S. Khan [14].*

Again the result of theorem (2.1) can be further generalized. In this case, the mapping T *is neither continuous nor satisfies the condition (1) but* T^m *for some positive integer* m *satisfies the same rational condition and contiuous,* T *still consumes a unique fixed point in* X.

Theorem 2.5: *Let T be continuous self map defined in 2- metric space* (X,d) *such that for some positive integer m, satisfies the condition*

$$d(T^m x,T^m y,a)\le \alpha\frac{d(x,T^m x,a)d(y,T^m y,a)+d(x,T^m y,a)d(y,T^m x,a)}{d(x,y,a)}$$

$$+\beta\frac{d(x,T^m y,a)[d(x,T^m x,a)d(y,T^m y,a)]}{d(x,y,a)+d(y,T^m y,a)+d(y,T^m x,a)}$$
$$+\gamma\frac{d(x,T^m y,a)d(y,T^m x,a)+d(y,T^m y,a)d(x,T^m y,a)}{d(x,T^m x,a)+d(y,T^m x,a)+d(y,T^m y,a)+d(x,T^m y,a)}$$
$$+\rho\frac{d(x,T^m x,a)d(x,T^m y,a)+d(y,T^m y,a)d(y,T^m x,a)}{d(x,T^m y,a)+d(y,T^m x,a)}$$
$$+\delta[d(x,T^m x,a)+d(y,T^m y,a)]+\eta[d(y,T^m x,a)+d(x,T^m y,a)]+\mu d(x,y,a)$$

for all $x,y\in X, x\neq y$ *and for* $\alpha,\beta,\gamma,\rho,\delta,\eta,\mu\in[0,1)$ *such that* $2\alpha+2\rho+4\delta+4\eta+2\mu<2$,

if T^m *is continuous then* T *has a fixed point in* X.

Proof. *By theorem (2.2), it is obvious that* T^m *has a unique fixed point* u *in* X
i.e $T^m(u)=u$. *Also*

$$T(u)=T(T^m u)=T^m(Tu)$$

From both relations we conclude that $T(u)=u$. *i.e* T *has a fixed point* u *in X. This completes the proof of theorem.*

Theorem 2.6: *Let* $\{T_n\}$ *be a sequence of mappings of 2- metric space* (X,d) *into itself. Let* x_n *be a fixed point of* $\{T_n\}$ $(n=1,2,\ldots)$ *and suppose* $\{T_n\}$ *converges uniformly to* T_0. *If* T_0 *satisfies the condition*

$$d(T_0x,T_0y,a)\le\alpha\frac{d(x,T_0x,a)d(x,T_0y,a)+d(y,T_0y,a)d(y,T_0x,a)}{d(x,T_0y,a)+d(y,T_0x,a)}$$
$$+\beta\frac{d(x,T_0y,a)[d(x,T_0x,a)+d(y,T_0y,a)]}{d(x,y,a)+d(y,T_0y,a)+d(y,T_0x,a)}+\gamma d(x,y,a)$$

for all $x,y\in X, x\neq y$ *and for* $\alpha,\beta,\gamma\in[0,1)$ *such that* $\alpha+\beta+\gamma<1$ *then* $\{x_n\}$ *converges to the fixed point* x_0 *of* T_0.

Proof. *From Theorem (2.1) and by given remarks conclude that* T_0 *has a unique fixed point satisfying the given rational expression. Let* $\varepsilon>0$ *be given, then there exists a natural number N such that*

$$d(T_nx,T_0x,a)<\frac{\varepsilon}{1-(\alpha+\beta+\gamma)}$$

For all $x\in X$ *and* $n>N$.

$$d(x_n,x_0,a)=d(T_nx_n,T_0x_0,a)\le d(T_nx_n,T_0x_n,a)+d(T_0x_n,T_0x_0,a)$$
$$\le d(T_nx_n,T_0x_n,a)+\alpha\frac{d(x_n,T_0x_n,a)d(x_n,T_0x_0,a)+d(x_0,T_0x_0,a)d(x_0,T_0x_n,a)}{d(x_n,T_0x_0,a)+d(x_0,T_0x_n,a)}$$
$$+\beta\frac{d(x_n,T_0x_0,a)[d(x_n,T_0x_n,a)+d(x_0,T_0x_0,a)]}{d(x_n,x_0,a)+d(x_0,T_0x_0,a)+d(x_0,T_0x_n,a)}+\gamma d(x_n,x_0,a)$$
$$=d(T_nx_n,T_0x_n,a)+\alpha\frac{d(x_n,T_0x_n,a)d(x_n,x_0,a)+d(x_0,x_0,a)d(x_0,T_0x_n,a)}{d(x_n,x_0,a)+d(x_0,T_0x_n,a)}$$
$$+\beta\frac{d(x_n,x_0,a)[d(x_n,T_0x_n,a)+d(x_0,x_0,a)]}{d(x_n,x_0,a)+d(x_0,x_0,a)+d(x_0,T_0x_n,a)}+\gamma d(x_n,x_0,a)$$

Such that

$$d(x_n,x_0,a)\le\frac{1}{1-(\alpha+\beta+\gamma)}d(T_nx_n,T_0x_n,a)<\varepsilon\quad for\ n>N.$$

This shows that $\{x_n\}$ *converges to* x_0 .

This completes the proof of the theorem.
Remark: *In the above theorem, if* $\beta = \gamma = 0$ *then we get theorem (2.2) of M. S. Khan [14].*

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